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SERIES IN SOME MITTAG-LEFFLER TYPE FUNCTIONS: THEOREMS FOR THEIR CONVERGENCE IN COMPLEX DOMAIN

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Abstract

In this paper Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems for series in some multi-index Mittag-Leffler functions are given.

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1. Introduction

The Mittag-Leffler (M-L) functions are natural extensions of the exponential function and trigonometric functions like the cos-function. They are defined in the whole complex plane $\mathbb{C}$ by the power series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0,\ \beta > 0), \qquad (1.1)$$

and their basic properties appeared back in the Bateman Project [3], Vol. 3, in a chapter devoted to "miscellaneous functions". These functions have been studied in details by Dzrbashjan [2]. The detailed properties of these functions can be found in the contemporary monographs [5], [6], and [18].

Recently a class of special functions of Mittag-Lefler type that are multi-index analogues of $E_{\alpha,\beta}(z)$ has been introduced and studied (see e.g. [7], [8]). The indices $\alpha := 1/\rho$, $\beta := \mu$ are replaced by two sets of multi-indices $\alpha \to (1/\rho_1, 1/\rho_2, ..., 1/\rho_m)$, and $\beta \to (\mu_1, \mu_2, ..., \mu_m)$.

Definition 1.1. Let $m > 1$ be an integer, $\rho_1, ..., \rho_m > 0$, $\mu_1, ..., \mu_m$ be arbitrary real (complex) numbers. By means of these "multi-indices", the *multi-index Mittag-Leffler functions* (multi-M-L f-s) are defined as:

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ \dots\ \Gamma\left(\mu_m+\frac{k}{\rho_m}\right)}. \tag{1.2}$$

The same functions, considered also by Luchko [10] and Yakubovich and Luchko [20] are called *Mittag-Leffler functions of vector index*. As proved in [8], the multi-index Mittag- Leffler functions (1.2) are entire functions of order ρ with $\frac{1}{\rho}=\frac{1}{\rho_1}+\dots+\frac{1}{\rho_m}$ and type $\sigma=\left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}}\dots\left(\frac{\rho_m}{\rho}\right)^{\frac{\rho}{\rho_m}}$.

In this work series of such kind of functions are considered, their domains of convergence are found and the series behaviour are studied on the boundary of these complex domains. In our previous papers ([13]-[16]) we studied series in systems of some representatives of special functions of fractional calculus (SF of FC) which are fractional indices analogues of the Bessel functions and also multi-index M-L functions (in the sense of Kiryakova [9]).

The proofs of some of the theorems can be found in [13]-[16] and the other proofs follow the lines of the proofs in our previous papers, for series in Bessel type functions.

2. Examples of multi-index Mittag-Leffler functions

A special case (for $m\geq 2$) is the generalized Lommel-Wright function with 4 indices ($\mu>0, q\in\mathbb{N}, \nu,\lambda\in\mathbb{C}$), introduced by de Oteiza, Kalla and Conde [12]:

$$\begin{aligned}J_{\nu,\lambda}^{\mu,q}(z)&=(z/2)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\,\Gamma(\nu+k\mu+\lambda+1)}\\&=(z/2)^{\nu+2\lambda}E^{(q+1)}_{(\mu,1,\dots,1),\,(\nu+\lambda+1,\,\lambda+1,\dots,\,\lambda+1)}\,(-(z/2)^2).\end{aligned} \tag{2.1}$$

This is an interesting example of a multi-index M-L function with arbitrary $m=q+1$. For $m=2$ the functions (1.2) is Dzrbashjan's M-L type functions from [1], denoted as $E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}$.

Some other interesting cases are given below.

Obviously for $q=1$, special function (2.1) turns into the generalization of the Bessel function $J_\nu(z)$, introduced by Pathak [17] (for details see [9]):

$$\begin{aligned}J_{\nu,\lambda}^{\mu}(z)&=(z/2)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)}\\&=(z/2)^{\nu+2\lambda}E^{(2)}_{(\mu,1),\,(\nu+\lambda+1,\,\lambda+1)}\,(-(z/2)^2).\end{aligned} \tag{2.2}$$

For particular choices of the other parameters λ and μ we obtain results for more special cases, e.g. the classical Bessel functions $J_\nu(z)$ and so called Bessel-Maitland function $J_\nu^\mu(z)$ (on the name E. Maitland Wright), introduced in [19].

3. Series in Mittag-Leffler type functions

We introduce auxiliary functions, associated with the Mittag-Leffler functions, namely:

$$\widetilde{E}_{0,\beta}(z) = 1; \quad \widetilde{E}_{n,\beta}(z) = \Gamma(\beta) z^n E_{n,\beta}(z), \tag{3.1.β}$$

$$\widetilde{E}_{\alpha,0}(z) = 1; \quad \widetilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z), \tag{3.1.α}$$

$$(n \in \mathbb{N}; \quad \beta > 0, \quad \alpha > 0)$$

and consider series in these functions in the complex plane, respectively of the forms:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\beta}(z)\,, \qquad (3.2.\beta) \qquad ; \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z)\,, \tag{3.2.α}$$

$$\sum_{n=0}^{\infty} a_n J_{n-2\lambda,\,\lambda}^{\mu,\,m}(z)\,, \tag{3.3}$$

$$(m \in \mathbb{N}, \quad \mu > 0, \quad \lambda \in \mathbb{C})$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Our main objective is to study the convergence of the series (3.2.β), (3.2.α) and (3.3) in the complex plane. Here we propose theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems for power series. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler functions (see for example, in [11]). Convergence theorems have also been obtained by the author - for series in Bessel functions and their Wright's 2 and 3-index generalizations, see the previous papers [13], [15].

4. Cauchy-Hadamard and Abel type theorems

First we give a theorem of Cauchy-Hadamard type for each of the above series.

Theorem 4.1. (of Cauchy-Hadamard type). *The domain of convergence of each one of the series* (3.2.β), (3.2.α), *(3.3) with complex coefficients a_n is the disk $|z| < R$ with the radius of convergence $R = 1/\Lambda$, where*

$$\Lambda = \limsup_{n\to\infty} (\, |a_n|\,)^{1/n}. \tag{4.1}$$

for the series (3.2.β), (3.2.α) *and there exists a nonnegative number $p \in \mathbb{N}_0$ such that*

$$\Lambda = 2^{-1} \limsup_{n\to\infty} (\, |a_n| \; |(\Gamma(\lambda + p + 1))^{\,m}\, \Gamma(n - \lambda + p\mu + 1)|^{-1})^{1/n}. \tag{4.2}$$

for the series (3.3) in Lommel-Wright functions.
The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case too, provided $1/\Lambda$ means ∞, respectively 0.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0.

Theorem 4.2. (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be the real number defined by Theorem 4.1, $0 < \Lambda < \infty$. Let $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. If $g(z;\beta)$, $h(z;\alpha)$, $j(z)$ are, respectively, the sums of the series* (3.2.β), (3.2.α), *(3.3) on the domain K, and these series converge at the point z_0 of the boundary of K, then:*

$$\lim_{z\to z_0} g(z;\beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\,\beta}(z_0), \quad (4.3.\beta); \quad \lim_{z\to z_0} h(z;\alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z_0); \quad (4.3.\alpha)$$

$$\lim_{z\to z_0} j(z) = \sum_{n=0}^{\infty} a_n J_{n-2\lambda,\lambda}^{\mu,\,m}(z_0), \tag{4.4}$$

provided $|z| < R$ and $z \in g_\varphi$.

The proofs of Theorem 4.1. and Theorem 4.2., using the specific properties of the considered functions, follow the lines of the analogous type theorems in [13]-[16].

5. (E, z_0) summations

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, ... \tag{5.1}$$

Note that each of the functions $\widetilde{E}_{n,\beta}(z), \widetilde{E}_{\alpha,n}(z)$ $(n \in \mathbb{N})$, $J_{n-2\lambda,\lambda}^{\mu,\,m}(z)$ being an entire function, not identically zero, has at most a finite number of zeros in the closed and bounded set $|z| \le R$. Moreover, due to proven asymptotic formulae ([14], [16]), only finite number of these functions may have some zeros at all except 0.

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, $\widetilde{E}_{n,\beta}(z_0) \neq 0$, $\widetilde{E}_{\alpha,n}(z_0) \neq 0$ and $J_{n-2\lambda,\lambda}^{\mu,\,m}(z_0) \neq 0$. For the sake of brevity, denote

$$J^*_{n,\lambda,\mu,m}(z;z_0) = \frac{J_{n-2\lambda,\lambda}^{\mu,m}(z)}{J_{n-2\lambda,\lambda}^{\mu,m}(z_0)}, E^*_{n,\beta}(z;z_0) = \frac{\widetilde{E}_{n,\beta}(z)}{\widetilde{E}_{n,\beta}(z_0)}, E^*_{\alpha,n}(z;z_0) = \frac{\widetilde{E}_{\alpha,n}(z)}{\widetilde{E}_{\alpha,n}(z_0)}. \tag{5.2}$$

Further, we introduce the following new notion of summability, related to series in M-L functions.

Definition 5.1. The series (5.1) is said to be (J, z_0)-*summable* (respectively (E_β, z_0), (E_α, z_0)-*summable*), if the series

$$\sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z;z_0), \tag{5.3}$$

respectively

$$\sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z;z_0), \quad (5.4.\beta); \qquad \sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z;z_0), \quad (5.4.\alpha)$$

converge in the disk $|z| < R$ and, moreover, there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z;z_0), \tag{5.5}$$

respectively

$$\lim_{z\to z_0}\sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z;z_0), \qquad (5.6.\beta); \qquad\qquad \lim_{z\to z_0}\sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z;z_0), \qquad (5.6.\alpha)$$

provided z remains on the segment $[0, z_0]$.

Remark 5.1. Every (J, z_0)-, (E_β, z_0)-, (E_α, z_0)-summation is regular, and this property is just a particular case of Theorem 4.2.

6. Tauberian type theorems

A Tauberian theorem is a statement that relates the Abel summability and the standard convergence of a number series by means of some assumptions imposed on the general term of the series under question. A classical result in this direction is given by Theorem 85 in Hardy [4].

In this paper we extend the validity of such type of assertion to series in Mittag-Leffler and Lommel-Wright functions, by means of the following theorem.

Theorem 6.1. (of Tauber type). *If the series* (5.1) *is* (J, z_0)*- summable (resp.* (E_β, z_0) (E_α, z_0)*), and*

$$\lim_{n\to\infty} na_n = 0, \qquad (6.1)$$

then it is convergent.

At first sight, it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeds to weaken it and to obtain the following stronger version of the Tauber theorem (see [4], Theorem 90).

A Littlewood generalization of the $o(n)$- version of the Tauber type theorem (Theorem 6.1) is given below.

Theorem 6.2 (of Littlewood type). *If the series (5.1) is* (J, z_0) *- summable, resp.* (E_β, z_0)*,* (E_α, z_0) *- summable, and*

$$a_n = O(1/n), \qquad (6.2)$$

then the series (5.1) converges.

Tauberian type theorems have also been given for summations by means of Bessel type functions by the author [13]-[15].

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